Simplified Calculation of System Outage Caused by Polarization-Mode Dispersion

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***Abstract*—Measuring system outage probability using known combinations of first- and second-order polarization-mode dispersion (PMD) is a powerful technique that has been difficult to implement due to the obstacle of approximating the joint probability density function (PDF) of PMD. We describe a closed-form approximation to this joint PDF in terms of elementary functions and assess its accuracy by comparison with a previous numerical integration and by comparing marginalizations of the approximation to known theoretical distributions. We then present a simple calculation of PMD-induced outage probability based upon measured performance.**

***Index Terms*—Monte Carlo methods, optical fiber communication, outage probability, polarization-mode dispersion (PMD).**

I. INTRODUCTION

POLARIZATION-MODE dispersion (PMD) can impose significant limitations upon the performance of a fiber channel, especially for older fibers and high bit rates. Direct measurement of the penalties induced by PMD is complicated by the fact that PMD within a narrow-band channel varies over time. In a widely accepted approach to the measurement of outage probability, penalties are measured using a PMD source that generates known combinations of first-order and second-order PMD (SOPMD) [1], [2]. These measurements are weighted using the joint probability density function (PDF) of PMD to yield a total outage probability as discussed in Section IV. The joint PDF has been shown to be scalable by a single parameter that characterizes the link PMD, e.g., the mean differential group delay (DGD) $\langle\tau\rangle$ [3]. Accordingly, the dimensionless variables $x = \mathrm{DGD}/\langle\tau\rangle$ and $y = |\mathrm{SOPMD}|/\langle\tau\rangle^2$ are used throughout this letter.

Calculation of outage probability has historically been difficult because no closed-form expression is known for the joint PDF $f_{\mathrm{PMD}}(x,y)$. Both numerical integration [4] and Monte Carlo simulation [5]–[7] have been used to calculate the joint PDF. In this letter, we describe a closed-form approximation to $f_{\mathrm{PMD}}(x,y)$ in terms of elementary functions of x and y (i.e., of normalized first- and second-order PMD), and show how it can be used to calculate outage probability.

Straightforward application of numerical integration software packages, e.g., MATLAB or Mathematica, has not yielded consistent solutions of the integrals describing the joint PDF (to the best of the author's knowledge). Although a detailed approach to numerical evaluation of the integrals has been published [4], these techniques require careful execution and significant effort. Likewise, a Monte Carlo simulation also requires careful construction and verification. Any person desiring to calculate a system outage probability for the first time might therefore expect to devote at least several days of effort to obtaining the joint PDF. In contrast, the approximation described in Section III can be coded and used in a few minutes. By eliminating the tedious numerical integration or modeling required to construct $f_{\mathrm{PMD}}(x,y)$, this approximation provides for convenient calculation of outage probability induced by PMD.

II. MONTE CARLO SIMULATION OF JOINT PDF USING IMPORTANCE SAMPLING

Monte Carlo techniques with importance sampling [5], [6] were used to build a two-dimensional histogram representing $f_{\mathrm{PMD}}(x,y)$. The most consistent results were obtained with Maxwell-distributed elements as described in [6], in which two parameters allow the simulation to be biased toward a targeted combination of first- and second-order PMD. We combined four simulations, each with a different bias toward unlikely events. The two simulations with the least bias toward unlikely events employed 20 Maxwell-distributed elements, while the other two simulations employed 40 elements to obtain better accuracy at high PMD. In all four cases the bias between first- and second-order PMD was uniformly distributed over its full range. Each simulation resulted in a two-dimensional discrete histogram; the four simulations were then combined using the maximum heuristic [7], i.e., by selecting, at each bin of the histogram, the simulation that contributed the most samples to that bin.

An approximation of $f_{\mathrm{PMD}}(x,y)$ valid over the domain $\{0 < x < 5.5, 0 < y < 11\}$ provides for the calculation of total outage probabilities from unity to less than 10^{-9}, a range more than adequate to characterize real systems that are often required to exhibit outage probabilities of the order of 10^{-5}. The histogram constructed by the Monte Carlo simulation covered this domain with a bin size of 0.05 in x and 0.1 in y. The combination of four simulations included a total of 33 million simulated links, but importance sampling yielded a reasonably smooth result over a dynamic range of approximately 10^{26}. A contour plot of the combined histogram is shown in Fig. 1, along with contours of a smooth least squares fit.

III. LEAST SQUARES FIT OF ELEMENTARY FUNCTIONS

A fit to the logarithm of the density function accommodates its large dynamic range and allows for the most accurate calculation of outage probability over a large range. We choose to search for an approximation $\tilde{f}_{\mathrm{PMD}}(x,y)$ in terms elementary

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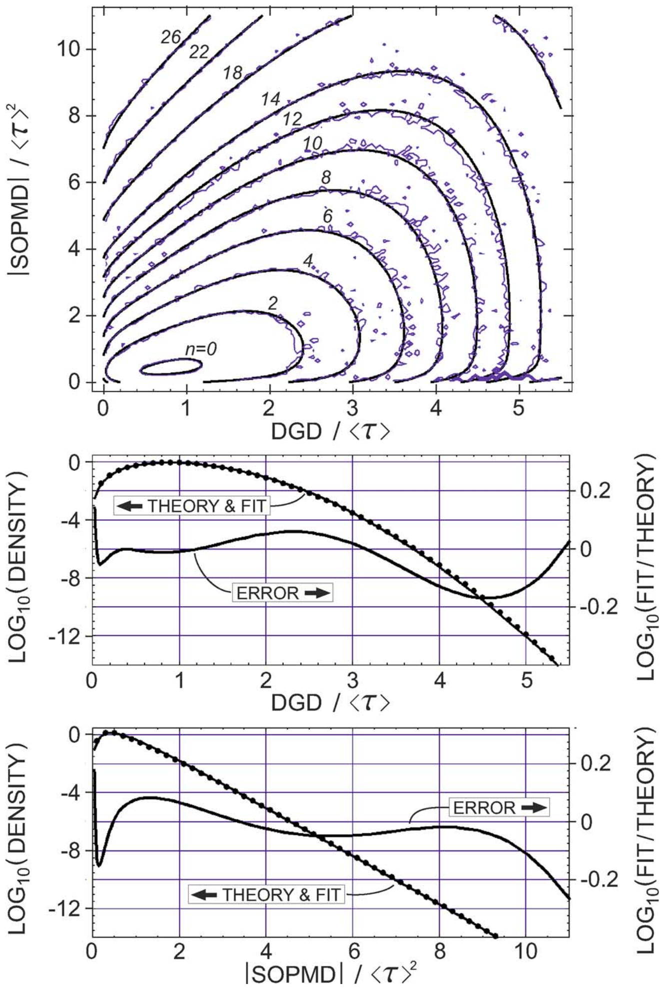

Fig. 1. (upper) Contour plot of the first- and second-order joint PDF of PMD. Contours are shown at 10^{-n}, with n indicated at each contour. The smooth least squares fit is shown along with the raw histogram contours. (center) Distribution of DGD; marginalization of fit compared to theory. (lower) Distribution of SOPMD amplitude; marginalization of fit compared to theory. Theoretical values are shown plotted at discrete points.

functions grouped into two basis vectors B_x and B_y, allowing a compact and numerically efficient expression. The functions are scaled to limit the range of weight coefficients. The approximation is

$$\tilde{f}_{\mathrm{PMD}}(x,y) = 10^{\wedge}[B_y^T(y) \cdot W \cdot B_x(x)] \qquad (1)$$

where

$$B_x(x) = \begin{bmatrix} (0.05+x)^{-0.2} \\ 1 \\ x \\ x^2 \\ \frac{x^3}{10} \end{bmatrix}, \quad B_y(y) = \begin{bmatrix} e^{-6y} \\ e^{-y} \\ 1 \\ y \\ y^2 \\ \frac{y^3}{100} \end{bmatrix}$$

and W is a matrix of 30 coefficients. In other words, $B_y^T(y) \cdot W \cdot B_x(x)$ expresses the base-10 logarithm of probability density at the point (x,y) on the normalized second-versus-first-order PMD plane. $B_y^T(y) \cdot W \cdot B_x(x)$ is a sum of 30 terms, each term a product of one function from B_x and one from B_y weighted by the corresponding element of W, where we have (2), as shown at the bottom of the page.

B_x and B_y were chosen by trial and error to achieve an adequate fit to the histogram with a small number of coefficients. Although these basis functions were found to yield a good fit (as shown in Fig. 1), other basis vectors might work as well and no physical significance should be ascribed to any of the functions. Adding higher order functions to the bases did not significantly improve the fit, due to the roughness of the histogram. The 30 coefficients of W were determined by a two-dimensional least squares fit to the histogram.

Contours of $\tilde{f}_{\mathrm{PMD}}(x,y)$ are shown in Fig. 2 for the case of a 3.257-ps mean DGD over a domain corresponding to $\{0 < x < 5.5,\ 0 < y < 11\}$. This allows direct comparison with a result calculated by numerical integration, shown in [4, Fig. 6]. The two results are very similar, with differences less than approximately $10^{0.1}$ over most of the plotted range. Larger differences occur in limited regions along the coordinate axes.

Although no exact expression of $f_{\mathrm{PMD}}(x,y)$ is available for comparison to the approximation $\tilde{f}_{\mathrm{PMD}}(x,y)$, the one-dimensional distributions of first-order and second-order PMD are known [3]. One measure of the accuracy of the fitted approximation can be obtained by comparing these known theoretical one-dimensional distributions to the two marginal distributions $f_{\mathrm{MARGIN,X}}(x) = \int_0^{14} \tilde{f}_{\mathrm{PMD}}(x,y)\,dy$ and $f_{\mathrm{MARGIN,Y}}(y) = \int_0^{7} \tilde{f}_{\mathrm{PMD}}(x,y)\,dx$. The lower plots of Fig. 1 show this comparison. The upper limits of integration $\{x < 7,\ y < 14\}$, while not infinite, were chosen to be large enough that small changes in these limits would not significantly affect the marginalizations, and at the same time small enough to avoid regions where the approximation clearly breaks down.

$$W = \begin{bmatrix} -6.84413 & 7.26557 & -3.75161 & 1.52517 & -2.05542 \\ 10.7254 & -17.2916 & 3.79351 & -0.886757 & 1.03959 \\ -13.0791 & 21.067548 & -4.33893 & 0.208947 & -0.657333 \\ 5.87094 & -13.2708 & 5.65587 & -1.31861 & 1.13062 \\ -0.912859 & 1.53578 & -0.798822 & 0.230175 & -0.218831 \\ 3.97736 & -6.91356 & 3.69591 & -1.13612 & 1.12773 \end{bmatrix} \qquad (2)$$

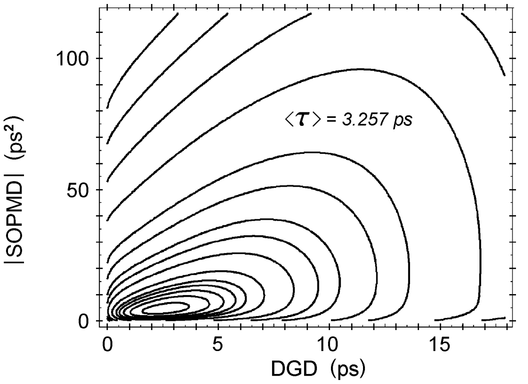

Fig. 2. Contour plot of the approximated first- and second-order joint PDF of PMD for a mean DGD of 3.257 ps. Contours are shown at 10^{-n}, with $n = \{1.5, 1.75, 2, 2.25, 2.5, 3, 4, 5, 6, 8, 10, 15, 20, 25, 30\}$.

IV. Application to Measurement of Outage Probability

The joint PDF given by (1) and (2) allows us to estimate total outage probability based upon measured failure rates. A PMD source that generates known combinations of first- and second-order PMD is used in place of the fiber link in an optical transmission system. In general, polarization scanners should be inserted before and after the PMD source to ensure that the principal states of polarization at both the input and output of the PMD source are varied over time relative to the transmitter and the receiver. At each PMD state we measure an estimate of the failure rate $U(x, y) = \Pr(\text{failure} \,|\, x, y)$. Failure can be judged in any way that is meaningful to the users of the system. In systems where PMD induces infrequent errors, $U(x, y)$ may be taken to be the fraction of errored seconds during a sufficiently long observation at the emulator state (x, y). Using the approximate joint PDF of PMD, the outage probability can then be calculated as an expected value using

$$P_{\text{outage}} = \Pr(\text{failure}) = \iint U(x, y)\tilde{f}_{\text{PMD}}(x, y)\, dx\, dy \tag{3a}$$

$$\lesssim B + \sum_{k=1}^{M} U_k \iint_{cellk} \tilde{f}_{\text{PMD}}(x, y)\, dx\, dy. \tag{3b}$$

Since the failure rate usually cannot easily be measured as a continuous function, we divide the (x, y) plane into a contiguous set of M measurement resolution cells, as shown in Fig. 3. $U(x, y)$ is measured as the PMD source is set to a sequence of states, each of which is represented by a point near the center of the corresponding cell. If we reference each cell with an index k and assume that $U(x, y) = U_k$ is approximately constant throughout the region of each cell, an approximate upper bound on the outage probability can be expressed as the sum (3b), where the background probability B is the integral of $\tilde{f}_{\text{PMD}}(x, y)$ over the untested region shown as the hatched area of Fig. 3. Alternatively, (3a) can be evaluated using an interpolated version of $U(x, y)$, but interpolation provides little benefit since $\tilde{f}_{\text{PMD}}(x, y)$ is a smooth function.

As an example, consider the measurement of Fig. 3, where the PMD source was set to combinations of DGD $= \{1, 4, 7, \ldots 28\}$ ps and $|\text{SOPMD}| = \{28, 84, 140, \ldots 252\}$ ps^2. To find the outage probability for a link with $\langle\tau\rangle = 8$ ps, we place the cell boundaries midway between the measured states, at $x = \{0, 2.5/8, 5.5/8, 8.5/8, \ldots 29.5/8\}$ and $y = \{0, 56/8^2, 112/8^2, 168/8^2, 224/8^2, 280/8^2\}$. Integration over the hatched area to the limits $\{x < 7, y < 14\}$ yields $B = 1.10 \times 10^{-6}$. Taking U_k for each cell to be the ratio of errored seconds to total seconds, the right-hand term of (3b) then evaluates to 2.00×10^{-7}, yielding an upper bound of 1.30×10^{-6} for the outage probability.

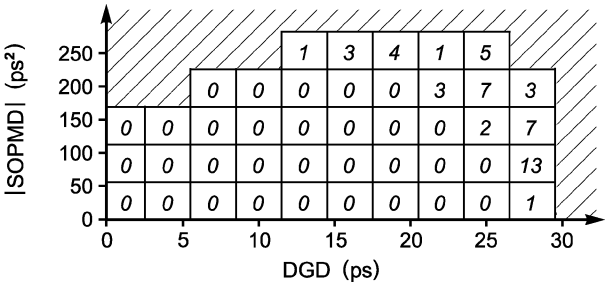

Fig. 3. Example of measurement resolution cells used to estimate the outage probability for a mean link DGD of 8 ps. Each cell displays the number of errored seconds measured over an observation time of 25 minutes during a test of a 43-Gb/s system incorporating an optical PMD compensator.

V. Conclusion

A useful closed-form approximation to the joint PDF of PMD in terms of elementary functions of first- and second-order PMD is given by (1) and (2). This approximation condenses the result of a large Monte Carlo simulation into 11 basis functions and 30 weights, allowing compact representation and communication of the joint PDF, and covering 16 decades of density variation. Due to its construction from elementary functions, the approximation is completely smooth and can be numerically integrated to precisely match the boundaries of any measurement resolution cell. Equation (3) expresses outage probability in terms of this approximation and experimentally measured quantities. The approximation (1) and (2) allows, for the first time, convenient and routine use of this approach.

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